Matrix: $A = [a_{ij}]_{m\times n}$; Square matrix: if $m=n$

Skew symmetric: $A = -A^T$; Symmetric: $A = A^T$

Identity matrix: $I_n = [a_{ij}]_{n\times n}$, $a_{ij} = 1$ for $i=j$; $a_{ij} = 0$, $i\neq j$

Diagonal matrix: $A = [a_{ij}]_{m\times n}$, $a_{ij} = 0\ \forall\ i\neq j$

Scalar matrix: $A = [a_{ij}]_{n\times n}$, $a_{ij} = 0\ \forall\ i\neq j$; $a_{ij} = a$, $i=j$

Idempotent matrix: $A^2 = A$; Nilpotent: $A^m = 0$, $A\neq 0$

Orthogonal matrix: $AA^T = I$; Transpose of matrix: $a_{ij} = a_{ji}$

Hermitian matrix: $A = (\bar{A})^T = A^\theta$; Skew Hermitian: $A = -A^\theta$

~~Invertibl~~ Invertible matrix: for $A\ \exists\ B$ s.t. $AB = BA = I$

Unitary matrix: $AA^\theta = A^\theta A = I$

Normal matrix: $AA^T = A^TA$ or $AA^\theta = A^\theta A$

Singular matrix: $A = [a_{ij}]_{n\times n}$ s.t. $|A| = 0$

Non singular: $A = [a_{ij}]_{n\times n}$ s.t. $|A|\neq 0$

Null or zero matrix: $A = [a_{ij}]_{m\times n}$, $a_{ij} = 0\ \forall\ i,j$

Properties of matrix: $A+B = B+A$; $A+(B+C) = (A+B)+C$

$A+0 = 0+A = A$, $A-A = 0$, $A+B = A+C \Rightarrow B = C$

$K(A+B) = KA+KB$; $(K_1K_2)A = K_1(K_2A) = K_2(K_1A)$; $IA = AI = A$

$A(BC) = (AB)C$; $(AB)^T = B^TA^T$; $(A\pm B)^T = A^T\pm B^T$; $(A^T)^T = A$

$k(A^T) = (kA)^T$; $(ABC)^T = C^TB^TA^T$

Trace: $tr(A) = \sum_{i=1}^{n} a_{ii}$; $tr(A+B) = trA + trB$

$tr(AB) = tr(BA)$

Determinant: $|A| = 0 \Rightarrow A$ is singular; $|A|\neq 0 \Rightarrow A$ is non-singular, $|AB| = |A||B|$, $|KA| = K^n|A|$,

$|Adj(A)| = |A|^{n-1}$; $|Adj(Adj(A))| = |A|^{(n-1)^2}$

Symmetric Matrix: $A = A^T \Rightarrow A$ is symmetric; $A = [a_{ij}]_{n}$
$A + A^T$ is symmetric; A, B are symmetric $\Rightarrow KA, KB,$
$A + B$ are symmetric;
All positive integral powers of symmetric matrix are symmetric; A, B are symmetric $\Rightarrow AB$ is symmetric $\Leftrightarrow AB = BA$; A, B are symmetric $\Rightarrow AB + BA$ is symmetric
All positive even powers of skew symmetric are symmetric

Skew symmetric: $A = -A^T \Rightarrow A$ is skew symmetric;
$A - A^T$ is skew symmetric; A, B are skew symmetric $\Rightarrow KA, KB, A + B$ are skew symmetric
All positive odd integral powers of a skew symmetric are skew symmetric;
A, B are symmetric $\Rightarrow AB - BA$ is skew symmetric
Every square matrix A can be expressed as uniquely as sum of a symmetric $\frac{1}{2}(A + A^T)$ and skew symmetric $\frac{1}{2}(A - A^T)$

Unitary matrix: $A^*A = AA^* = I \Rightarrow A$ is unitary $\Rightarrow$ A^T and A^{-1} are unitary; A, B are unitary $\Rightarrow AB$ and BA are unitary

Invertible matrix: For $A \ \exists B$ s.t. $AB = BA = I$; $(AB)^{-1} = B^{-1}A^{-1}$
A, B are invertible $\Rightarrow AB$ is invertible but $A + B$ may not be invertible; A invertible $\Leftrightarrow A^T$ is invertible

Idempotent matrix: $A^2 = A \Rightarrow A$ is idempotent
A, B are idempotent $\Rightarrow AB$ is idempotent $\Leftrightarrow AB = BA$
$A + B$ is idempotent $\Leftrightarrow AB = BA = 0$; If $AB = A, BA = B \Rightarrow A^2 = A, B^2 = B$
A is idempotent, $A + B \Rightarrow AB = BA = 0$ & $B^2 = B$

Q. Define symmetric and skew symmetric matrices, also define Hermitian and skew Hermitian matrices.

Ans. Symmetric matrix: A square matrix $A=[a_{ij}]$ is said to be symmetric if its (i,j)th element is same as its (j,i)th element i.e. $a_{ij}=a_{ji}\ \forall i,j$

Skew Symmetric matrix: A square matrix $A=[a_{ij}]$ is said to be skew symmetric if the (i,j)th element of A is the negative of (j,i)th element of A i.e. $a_{ij}=-a_{ji}$ for all i,j

Hermitian matrix: A square matrix $A=[a_{ij}]$ is said to be Hermitian if the (i,j)th element of A is equal to the conjugate complex of the (j,i)th element of A i.e. $a_{ij}=-\bar{a}_{ji}$ for all i,j

Skew Hermitian matrix: A square matrix A $=[a_{ij}]$ is said to be skew Hermitian if the (i,j)th element of A is equal to the negative of the conjugate complex of the (j,i)th element of A i.e. $a_{ij}=-\bar{a}_{ji}$ for all i and j

Q. Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew symmetric matrix.

Ans. Let A be any square matrix, we can write
$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')=P+Q$ where
$\frac{1}{2}(A+A')=P$ and $\frac{1}{2}(A-A')=Q$

we have $P'=\left(\frac{1}{2}(A+A')\right)'=\frac{1}{2}(A+A')'$
$=\frac{1}{2}(A'+(A')')=\frac{1}{2}(A'+A)=\frac{1}{2}(A+A')=P$

$\therefore$ P is symmetric matrix

Again $Q'=\left(\frac{1}{2}(A-A')\right)'=\frac{1}{2}(A-A')'=\frac{1}{2}(A'-(A')')$
$=\frac{1}{2}(A'-A)=-\frac{1}{2}(A-A')=-Q$

Therefore Q is a skew symmetric matrix.

Representation is unique: let $A=R+S$ be another such representation of A, where R is symmetric and S is skew symmetric

Then to prove $R=P$ and $S=Q$

we have $A' = (R+S)' = R'+S' = R-S \because R'=R, S'=-S$

$\therefore A+A' = 2R$ and $A-A' = 2S$

This gives $R = \frac{1}{2}(A+A')$ and $S = \frac{1}{2}(A-A')$

Thus $R=P$ and $S=Q$

Therefore the representation is unique.

Q Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew Hermitian matrix.

Ans If A is any square matrix, then $A+A^{\theta}$ is a Hermitian matrix and $A-A^{\theta}$ is a skew Hermitian matrix. Therefore $\frac{1}{2}(A+A^{\theta})$ is a Hermitian and $\frac{1}{2}(A-A^{\theta})$ is a skew Hermitian matrix. Now we have

$$A = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta}) = P+Q$$

Where P is the Hermitian and Q is skew Hermitian. Thus every square matrix can be expressed as the sum of Hermitian and skew Hermitian matrix.

Let, $A=R+S$ be another representation of A, where R is Hermitian and S skew Hermitian, then $A^{\theta} = (R+S)^{\theta} = R^{\theta}+S^{\theta} = R-S \because R^{\theta}=R, S^{\theta}=-S$

$\therefore R = \frac{1}{2}(A+A^{\theta}) = P$ and $S = \frac{1}{2}(A-A^{\theta}) = Q$

Thus the representation is unique.

Q Show that all positive integral powers of a symmetric matrix are symmetric

Ans Let A be a symmetric matrix of order n. then $A^m = AAA \cdots A$ upto m times, m is a positive integer

Now $(A^m)' = (AA \cdots A)' = A'A'A' \cdots A'$ upto m times

$= AAA \cdots A = A^m \because A' = A$

Hence A^m is also a ~~symmet~~ symmetric matrix.