B.Sc Physics (H), Part - I

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Non Inertial Frame and Pseudo Force

When we consider particle motion or the motion of a system in an accelerated frame or non inertial frame, the particle or the system as a whole, will experience a force in opposite to the direction of acceleration of the frame. This force appeared in non inertial frame is called 'pseudo force'.

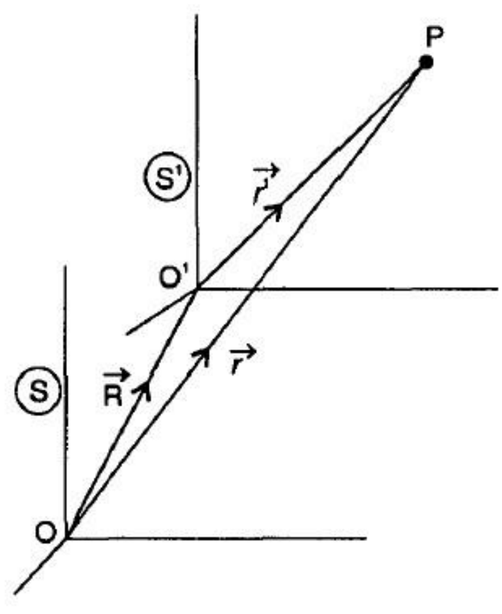
For the presence of such 'pseudo force', the effect of force acting on the particle will change than that in inertial frame. It can easily be shown that if $\vec{F}$ be the force applied on the particle in non inertial frame, the effective force acting on it will be

$$\vec{F}_{eff} = \vec{F} + \left(-\vec{F}_P\right) = \vec{F} + \vec{F}_O$$

Where, $\vec{F}_O = -\vec{F}_P$ is the pseudo force appearing in non inertial frame. The nature of this pseudo force entirely depends on the mode of acceleration of the frame. We can clear this point in the following cases—

(i) Case – I : *Pseudo Force in non inertial frames having translational accelerated motion*:

Consider two frames S and S′ when S is an inertial frame and S′ is non inertial or accelerated frame having acceleration $\vec{f}$ in its translational motion with respect to S. $\vec{R}$ is the instantaneous position vector of the point O′ of S′ with respect to the origin 'O' of S frame.



So if $\vec{r}$ and $\vec{r}'$ be the respective position vector of the instantaneous position P of a moving particle then from fundamental idea of vector algebra.

$$\vec{r} = \vec{R} + \vec{r}\,'$$

$$\Rightarrow \quad \vec{r}\,' = \vec{r} - \vec{R}$$

$$\therefore \quad \frac{d^2\vec{r}\,'}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \frac{d^2\vec{R}}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2\vec{r}\,'}{dt^2} = m\frac{d^2\vec{r}}{dt^2} + m\left(-\frac{d^2\vec{R}}{dt^2}\right)$$

$$\Rightarrow \quad \vec{F}_{eff} = \vec{F}' = \vec{F} + m\left(-\vec{f}\right) = \vec{F} + \left(-m\vec{f}\right)$$

$$\therefore \quad \vec{F}_{eff} = \vec{F} + \left(-\vec{F}_P\right).\left(\vec{F} = m\frac{d^2\vec{r}}{dt^2}\right)$$

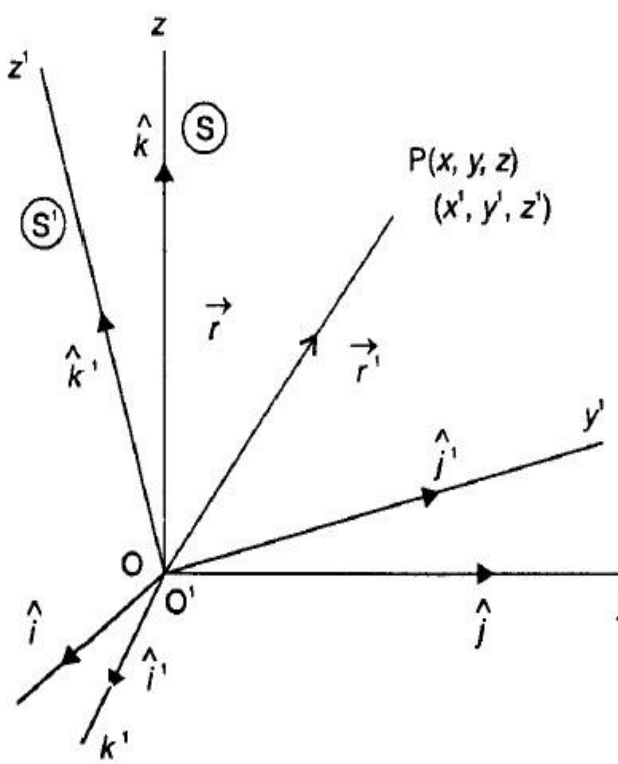
= actual force applied from outside.

where, $-\vec{F}_P = -m\vec{f}$ is the pseudo force acting on that particle considered in non-inertial frame.

This pseudo force sharply acts in direction opposite to the acceleration of the frame.

(ii) Case-II : *Pseudo Force in non inertial rotating frame* :

Here we consider that 'S' (oxyz) is an inertial frame where as 'S'' (O′ X′ Y′ Z′) is non inertial rotating frame having origin coinciding with that of 'S' frame. It is obvious that the basis unit vectors $\left(\hat{i}, \hat{j}, \hat{k}\right)$ of S-frame are fixed where as the other set of unit vectors $\left(\hat{i}', \hat{j}', \hat{k}'\right)$ changes their orientation w.r.t. time t during rotation of S′-frame.



So if $\vec{r}$ and $\vec{r}\,'$ be the respective positions of same instanteneous* position 'P' of the moving particle relative to S and S′ frame then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r}' = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$

Also for this two frames S and S′,

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \text{ in S frame or fixed frame.}$$

$$\text{and also } \left(\frac{d\vec{r}'}{dt}\right)_{\text{rot}} = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}'$$

When $\left(\hat{i}', \hat{j}', \hat{k}'\right)$ are itself fixed w.r.t. rotational frame S′ and these are rotating unit vectors when viewed in S-frame.

$$\therefore \quad \left(\frac{d\vec{r}'}{dt}\right)_{\text{fixed}} = \frac{dx'}{dt}\hat{i}' + \frac{dy'}{dt}\hat{j}' + \frac{dz'}{dt}\hat{k}' + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$\therefore \quad \left(\frac{d\vec{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{r}'}{dt}\right)_{\text{rot}} + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$\text{But we have * } \vec{v} = \frac{d\vec{r}}{dt} = \vec{w} \times \vec{r}$$

$$\therefore \quad \frac{d\hat{i}'}{dt} = \vec{w} \times \hat{i}', \ \frac{d\hat{j}'}{dt} = \vec{w} \times \hat{j}', \ \frac{d\hat{k}'}{dt} = \vec{w} \times \hat{k}'$$

$$\text{so,} \quad \left(\frac{d\vec{r}'}{dt}\right)_{\text{Fixed}} = \left(\frac{d\vec{r}'}{dt}\right)_{\text{rot}} + x'\left(\vec{w} \times \hat{i}'\right) + y'\left(\vec{w} \times \hat{j}'\right) + z'\left(\vec{w} \times \hat{k}'\right)$$

$$= \left(\frac{d\vec{r}'}{dt}\right)_{\text{rot}} + \vec{w} \times \left(x'\hat{i}' + y'\hat{j}' + z'\hat{k}'\right)$$

$$\therefore \quad \left(\frac{d\vec{r}'}{dt}\right)_{\text{Fixed}} = \left(\frac{d\vec{r}'}{dt}\right)_{\text{rot}} + \vec{w} \times \vec{r}'$$

$$\text{and we usually have,** } \left(\frac{d}{dt}\right)_{\text{Fixed}} = \left(\frac{d}{dt}\right)_{\text{rot}} + \vec{w}x.$$

This is the relation between time variation operators in fixed frame S(inertial) and rotating frame S′ (non-inertial).

Since, in our case, $\vec{r}' = \vec{r} = \overrightarrow{OP} = \overrightarrow{O'P}$

so, we should have,

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{Fixed} = \left(\frac{d}{dt}\right)_{Fixed}\left(\frac{d}{dt}\right)_{Fixed}(\vec{r})$$

$$= \left[\left(\frac{d}{dt}\right)_{rot} + \vec{w}\times\right]\left[\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{w}\times\vec{r}\right]$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + \vec{w}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{w}\times\left(\frac{d\vec{r}}{dr}\right)_{rot} + \vec{w}\times(\vec{w}\times\vec{r})$$

Where $\vec{w}$, the rotational velocity is time independent.

$$\therefore\ m\left(\frac{d^2\vec{r}}{dt^2}\right)_{Fixed} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + 2m\vec{w}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} + m\vec{w}\times(\vec{w}\vec{r})$$

But if $\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{rot}$ = velocity of moving particle in rotating frame

then, $$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{Fixed} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + 2m\vec{w}\times\vec{v} + m\vec{w}\times(\vec{w}\times\vec{r})$$

$$\Rightarrow \qquad m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{Fixed} + (-2m\vec{w}\times\vec{v}) + [-m\vec{w}\times(\vec{w}\times\vec{r})]$$

$$\therefore \qquad \vec{F}_{eff} = \vec{F} + \vec{F}_1 + \vec{F}_2$$

where $\vec{F}_{eff} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot}$ = effective force on particle in rotating non-inertial frame.

$$\vec{F} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{Fixed}}$$ = Actual force acting on particle in fixed inertial frame.

$\vec{F}_1 = \vec{F}_C = -2m(\vec{w}\times\vec{v})$ = Pseudo force which is commonly known as 'coriolis force'.* and $\vec{F}_2 = \vec{F}_C = -m\overline{w}\times(\vec{w}\times\vec{r})$ = Pseudo force which is popularly known as 'centrifugal force'.

So we see that in a rotating non inertial frame, two pseudo forces will appear, one of which is called coriolis force and other is centrifugal force. The action of Ist one is conditional and it is not effective for a rest particle w.r.t. rotating frame, where as the 2nd force always acts in rotating frame.

Effect of rotation of earth on acceleration due to gravity

Neglecting earth's translational motion around sun, if we only consider earth's rotation then we can show that the acceleration due to gravity at any place on earth surface except at pole, will change effectively due to earth's rotation.

Let us now consider two frames S and S′ when the frame S is inertial and is taken at the center of earth and the non-inertial frame S′ at any point on earth surface. Now if for any particle motion with velocity $\vec{v}=\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}}$ as measured from that frame S′ on earth surface, we take its weight $m\vec{g}$ directed towards the center of earth and any real force (actual) $\vec{F}$ acting on it, then the effective force on it as seen from the surface of earth due to earth's rotation, will become,

$$\vec{F}_{\text{eff}} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{rot}}$$

$$= m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{Fixed}} + [-m\vec{w}\times(\vec{w}\times\vec{r})] + [-2m\vec{w}\times\vec{v}]$$

$$\Rightarrow \quad \vec{F}_{\text{eff}} = (\vec{F}+m\vec{g}) + [-m\vec{w}\times(\vec{w}\times\vec{r})] + [-2m(\vec{w}\times\vec{v})]$$

$$= \vec{F} + m[g - \vec{w}\times(\vec{w}\times\vec{r})] - 2m(\vec{w}\times\vec{v})$$

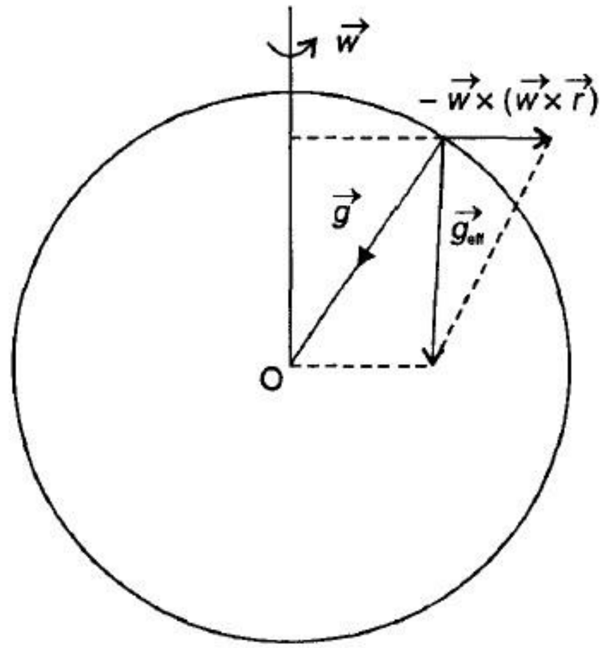
$$= \vec{F} + m\vec{g}_{\text{eff}} - 2m(\vec{w}\times\vec{v})$$

where $\vec{g}_{\text{eff}} = \vec{g} - \vec{w} \times (\vec{w} \times \vec{r})$ is the effective gravitational acceleration of the place on earth surface. This is the resultant of actual gravitational acceleration $\vec{g}$ and centrifugal acceleration due to earth's rotation $-\vec{w} \times (\vec{w} \times \vec{r})$ and as a result we have the nature of this effective gravitational acceleration as,

$$\vec{g}_{\text{eff}} = \vec{g} \text{ at pole}$$

$$= \vec{g} - \vec{w} \times (\vec{w} \times r)$$

at any place on earth surface other than at equator (not directed towards the center of earth).



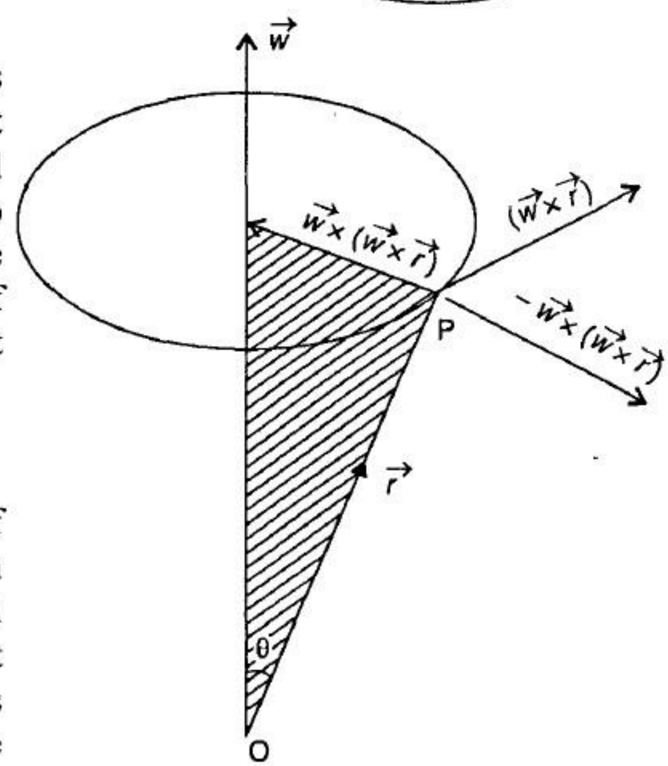
$= \vec{g} - \vec{w} \times (\vec{w} \times \vec{R})$ at equator (directed towards the center of earth.)

If we now be interested to calculate the maximum effect of this centrifugal acceleration on actual gravitational acceleration, then we will see that for earth radius R = 6400 km and angular velocity for earth rotation $w = \frac{2\pi}{86400}$ rad/sec = 7.29×10^{-5} rod/sec, the maximum value of centrifugal acceleration at equator is

$$a_2 = w^2.R = 3.4 \times 10^{-2}\ m/\text{sec}^2$$

which is about 0.2% of earths gravitational acceleration. So at equator the reduction of actual gravitational acceleration due to centrifugal acceleration will be maximum and the effective value of the gravitational acceleration at equator will be.

$$g_{\text{eff}} = g - w^2R = 9.81 - 0.034$$
$$= 9.776 \text{ m/sec}^2$$

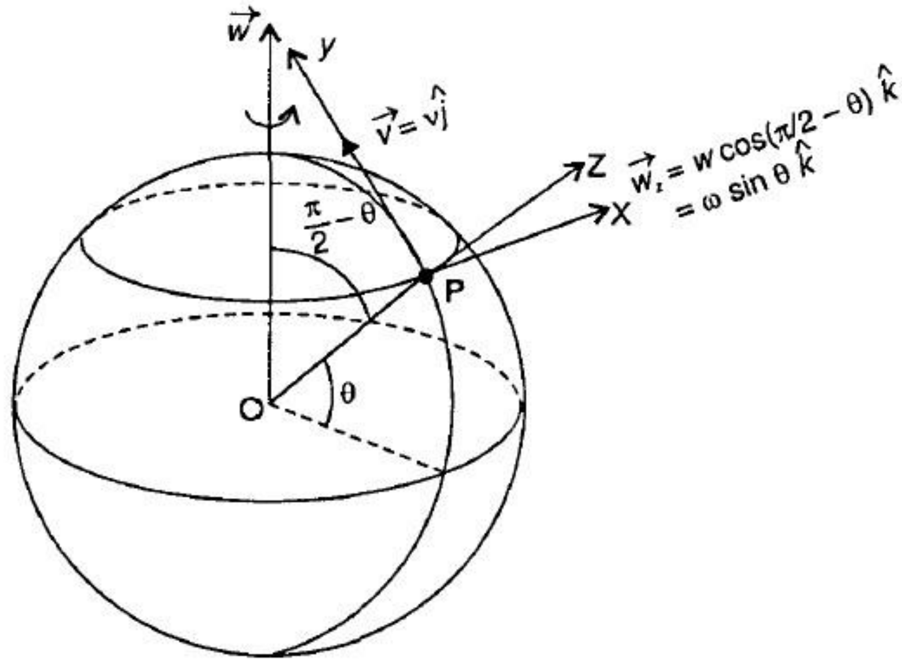


It is a fact that for rotation of frame, the centripetal acceleration will act in opposite to the centrifugal acceleration towards the center at which inertial rest frame exist. This fact is also true for any particle rotation above a fixed point.

Effect of Coriolis Force on a particle moving on the surface of earth

Here we are now interested about the effect of coriolis force on a particle moving on earth surface. We have from our earlier discussion that due to earth rotation, the coriolis force acting on particle moving with velocity $\vec{v}$ w.r.t. earth surface is

$\vec{F}_e = -2m(\vec{w} \times \vec{v})$



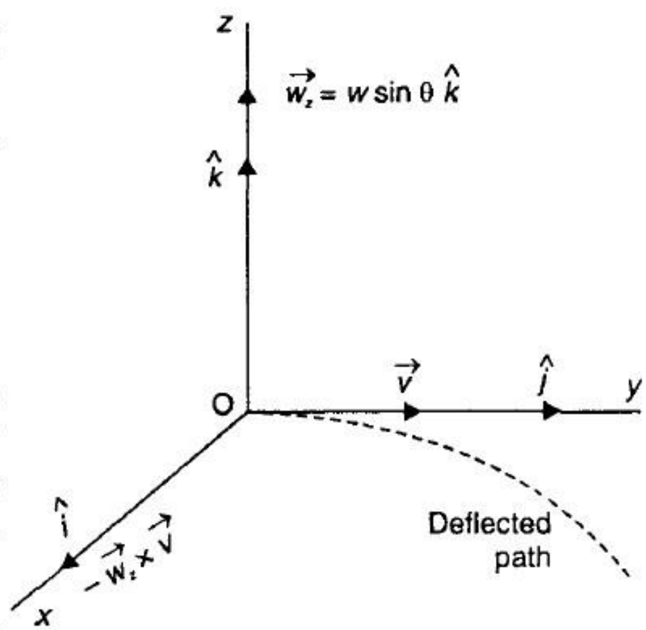
which will be at right angle to the plane of $\vec{v}$ and $\vec{w}$.

Now we consider particle motion at P on earth surface having a latitude θ, with velocity $\vec{v}$ along y-axis.

Here for the system OXYZ of S′ frame taken at P, the plane XY is horizontal plane w.r.t. earth surface at P and Z is vertical axis to the same point P, the effective component of earth's angular velocity along Z axis is

$$w_z = w\cos\left(\frac{\pi}{2} - \theta\right) = w\sin\theta .$$

So for particles velocity $\vec{v}$ along y-axis the coriolis acceleration will be $-2(\vec{w}_z \times \vec{v})$ and its magnitude is $2w_z v$ $= 2w \sin\theta\, v$.



This is a fact that for such acceleration in transverse direction to the particle motion, the particle will deviate from its original path (along y-axis) in transverse direction (in XY plane) and will finally move along the deflected path in XY plane as shown in figure.

The maximum effect of this transverse coriolis acceleration will occur at any pole and it is

$$a_c = 2\,w \sin\frac{\pi}{2}.v = 2wv = 2 \times 7.29 \times 10^{-5} v$$

$$\therefore \quad a_c = 1.458 \times 10^{-4} v.$$

When, v is the velocity in horizontal plane. If we now take a typical example of a particle moving horizontally on earth surface with velocity 1km/sec or 3600 km/hr, the magnitude of coriolis acceleration will become $a_c = 1.458 \times 10^{-4} \times 10^5 = 14.58$ cm/hr.^{2}

or, $a_c \approx 0.15$ m/sec^2

which is about 0.015 g.

We can now conclude that although the magnitude of coriolis acceleration is small, it plays an important role in many phenomenon in the earth. It is important to take into consideration the effects of the coriolis acceleration in the flight of missiles, the velocity and the time of flight which are considerably large.

Here for particle moving with velocity $\vec{v}$ in the horizontal plane on earth surface, we can immediately find out the angle of deflection for particle motion along the duration t as

$$\alpha = \frac{\begin{array}{c}\text{distance travelled in time } t \text{ in the deflected direction} \\ \text{(for sufficiently small deflection)}\end{array}}{\text{distance travelled in time } t \text{ in the direction of projection}}$$

$$= \frac{1}{2}\frac{(2w\sin\theta.v)t^2}{vt} = w\sin\theta.t$$

Thus, on the north pole ($\theta = 90°$), $\alpha = wt$ which is maximum and for any $t = 180$ sec.

$\alpha = 7 \times 10^{-3}$ radian $= 0.04°$ which is found to be quite small, but it assumes considerably important in guided missiles.

Effect of Coriolis force on a particle falling freely under gravity

We consider a particle which is falling freely (neglecting air resistance) from height 'h' on earth surface and in this case, due to coriolis acceleration only the acceleration of particle will become

$$\vec{a} = \vec{g} - 2\vec{w} \times \vec{v}$$

When $\vec{w}$ is angular velocity for earth rotation, $\vec{a}$ and $\vec{v}$ both are measured in rotating frame on earth surface.

Now from our previous topic of discussion, we have, at latitude θ on earth surface, where we have considered the falling of particle in $-\hat{k}$ direction (along -*ve* z direction),

$$w_x = v,\ w_y \cos\theta,\ w_z = w\sin\theta$$

Also, for particle motion along $-ve$ z direction,

$$\dot{x} \approx 0,\ \dot{y} \approx 0,\ \dot{z} = -gt$$

Again, for quite small deflection produced by coriolis force the components of the acceleration will be

$$\begin{aligned} a_x &= \ddot{x} = \left|-2(\vec{w} \times \vec{v})\right| = \left|-2(-\omega \cos\theta\, gt)\right| \\ &= 2w\, gt \cos\theta \\ a_y &= \ddot{y} = 0 \\ a_z &= \ddot{z} = -g \end{aligned}$$

Here the acceleration a_x along X direction is due to coriolis force and we thus get

$$\ddot{x} = \frac{d^2x}{dt^2} = 2wgt\cos\theta \Rightarrow x = \frac{1}{3}wgt^3\cos\theta$$

$$\ddot{z} = \frac{d^2z}{dt^2} = -g \Rightarrow z = z_0 - \frac{1}{2}gt^2$$

when, $t = 0$, $z = z_0$, $x = 0$, $y = 0$ is the initial position, with initial velocities $\dot{x}(0) = 0 = \dot{z}(0)$.

Since, the time of fall from height 'h' is

$t = \sqrt{\dfrac{2h}{g}}$, the deflection of a particle towards the east when it is dropped from rest is

$$\delta = \frac{1}{3}w\cos\theta\left(\frac{8h^3}{2}\right)^{1/2}.$$

This is called 'eastward effect' due to the effect of coriolis force on a freely falling body. As for example, if the particle be dropped from a height of 100 m from rest at latitude $\theta = 45°$, it will be deflected by about 1.55×10^{-2}m toward east in northen hemisphere.