Q. Define countable and Uncountable sets.

Ans Denumerable sets: let $N=\{1,2,3,\cdots\}$ be a set of natural numbers.

A set is said to be a denumerable set if $A \sim N$

i.e. a set A is called denumerable if its elements can be put to one-to-one correspondence with the set of natural numbers.

Thus $A=\{a_1, a_2, \cdots, a_n, \cdots\}$ is denumerable since $A \sim N$ under the mapping $f(n)=a_n$

A denumerable set is also called enumerable or countably infinite

Countable set: A set is called countable if it is finite or denumerable

Non-denumerable set: A set A is called a non-denumerable set if A is an infinite set and A is not equivalent to N. It is also called uncountable set or non-countable set

Q. If A_i is countably infinite set for $i=1,2,3,\cdots,n$ then $\bigcup_{i=1}^{n} A_i$ is countably infinite.

Ans Let $A=\bigcup_{i=1}^{n} A_i$, and elements of A_i can be displayed as follows

$$A_1 = a_{11}, a_{12}, a_{13} \cdots \cdots, a_{1n}, \cdots$$
$$A_2 = a_{21}, a_{22} \quad a_{23} \cdots \quad a_{2n}, \cdots$$
$$\cdots$$
$$A_n = a_{n1} \quad a_{n2} \quad a_{n3} \cdots \quad a_{nn} \cdots$$

Then Collecting the elements vertically columnwise from the above n sets $A=\bigcup_{i=1}^{n} A_i=\{a_{11}, a_{21}, a_{31}, \cdots a_{n1}, a_{12}, a_{22}, a_{33}, \cdots, \cdots\}$

writting $a_{11}=x_1, a_{21}=x_2, \cdots$

we can write A in the form of a sequence

$A=\{x_1, x_2, x_3, \cdots, x_n, x_{n+1}, \cdots\}$

This proves that A is enumerable.

Note: If A and B are enumerable sets then $A \cup B$ is also enumerable

Q Countable union of countable sets is countable ⊛
If A_i is countably infinite set for $i=1,2,3 \ldots$
then $\bigcup_{i=1}^{\infty} A_i$ is countably infinite.

As since A_i is enumerable for each i, therefore the elements of A_i can be exhibited as follows

$A_1 : a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n} \cdots$
$A_2 : a_{21} \quad a_{22} \quad a_{23} \quad \cdots \quad a_{2n} \cdots$
$\cdots$
$A_n : a_{n1} \quad a_{n2} \quad a_{n3} \quad \cdots \quad a_{nn} \cdots$
$\cdots$

we collect the elements of the array diagonal wise and thus we arrange them in a sequ.
Thus we form $A = \bigcup_{i=1}^{\infty} A_i = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots\}$
$= \{b_1, b_2, \ldots, b_k, \ldots\}$

where $k = \{1+2+3+\cdots+(m+n-2)\}+n$, $m+n \geq 2$
Thus A is expressible in the form of a sequence of distinct elements and hence A is enumerable.

Q. If A and B are countable set then $A \times B$ is also countable.

As let $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$ then
$A \times B = \{(a_1, b_1), (a_1, b_2), \ldots (a_2, b_1), (a_2, b_2), \ldots\}$
Now the elts of $A \times B$ can be arranged as follows
$(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots, (a_1, b_n) \ldots$
$(a_2, b_1), (a_2, b_2), (a_2, b_3), \ldots, (a_2, b_n) \ldots$
$\cdots$
$(a_m, b_1), (a_m, b_2) (a_m, b_3) \ldots (a_m, b_n) \ldots$
For each fixed index i, let us write
$A_i = \{(a_i, b_1), (a_i, b_2), \ldots\} = \{x_{i1}, x_{i2}, \ldots\}$ where x_{ip} stand for (a_i, b_p)
Then we see that (i) $A \times B = \bigcup A_i$
(ii) $A_i \cap A_j = \phi$ for $i \neq j$ (iii) A_i is enumerable for all i
from (ii) and (iii) we find that $\bigcup_{i=1}^{\infty} A_i$ is enumerable
Hence $A \times B$ is enumerable.