Q. PT The relation "congruence modulo n" is the set of integers is an equivalence relation

Ans. we define a relation R on the set of integers I by aR_b to mean $a-b$ is divisible by n for all $a,b \in I$ i.e.

$a \equiv b \pmod{n}$

To prove R is an equivalence relation

$a-a = 0 \cdot n \Rightarrow a \equiv a \pmod{n} \Rightarrow aRa$, R is reflexive

If $a \equiv b \pmod{n}$ we have $a-b = nq,\ q \in I$.

$\therefore b-a = n(-q) \Rightarrow b \equiv a \pmod{n} \Rightarrow bRa$, R is Symmetric

If aRb and bRc means

$a-b = nq$ and $b-c = nr$ where $q, r \in I$.

$a-c = a-b+b-c = nq+nr = n(q+r)$

$q+r$ is also integer. ~~Thus~~

Hence $a \equiv c \pmod{n} \Rightarrow aR_c$, R is transitive.

Thus R is an equivalence relation

Q. If R and R' are two equivalence relations in a set A, show that $R \cap R'$ is an equivalence relation but $R \cup R'$ may not be equivalence relation in A.

Ans. Since R and R' are relations in a set A

$\therefore R \subseteq A \times A$ and $R' \subseteq A \times A \therefore R \cap R' \subseteq A \times A$

Hence $R \cap R'$ is a relation in the set A.

Now we shall prove that $R \cap R'$ is an equivalence relation.

Since R and R' are equivalence relation in A $\therefore$ they are reflexive

$\therefore$ for all $x \in A$ we have $(x,x) \in R,\ (x,x) \in R'$

$\Rightarrow (x,x) \in R \cap R' \Rightarrow R \cap R'$ is reflexive

Let $(x,y) \in R \cap R' \Rightarrow (x,y) \in R$ and $(x,y) \in R'$

$\Rightarrow (y,x) \in R$ and $(y,x) \in R'$

$\Rightarrow (y,x) \in R \cap R'$

$\therefore R \cap R'$ is symmetric

Let $(x,y),(y,z) \in R \cap R'$

$\Rightarrow (x,y),(y,z) \in R$ and $(x,y),(y,z) \in R'$

$\Rightarrow (x,z) \in R$ and $(x,z) \in R'$, R, R' are transitive

$\Rightarrow (x,z) \in R \cap R'$ $\therefore$ $R \cap R'$ is transitive

Hence $R \cap R'$ is an equivalence relation

Now for Union

$R \cup R'$ is a subset of $A \times A$ and hence $R \cup R'$ is a relation in A

Since R and R' are reflexive

$\therefore (x,x) \in R, (x,x) \in R' \Rightarrow (x,x) \in R \cup R'$

$R \cup R'$ is reflexive

Let $(x,y) \in R \cup R' \Rightarrow (x,y) \in R$ or $(x,y) \in R'$

$\Rightarrow (y,x) \in R$ or $(y,x) \in R'$ $\because R, R'$ are symmetric

$\Rightarrow (y,x) \in R \cup R'$, $R \cup R'$ is symmetric

If R and R' are transitive on a set A then

let $A = \{1,2,3\}$ and $R = \{(1,1),(1,2)\}$ and

$R' = \{(2,2),(2,3)\}$ $\therefore R \cup R' = \{(1,1),(1,2),(2,2),(2,3)\}$

$(1,2),(2,3) \in R \cup R'$ but $(1,3) \notin R \cup R'$

so $R \cup R'$ is not transitive.

Thus $R \cup R'$ is not an equivalence relation.

Q. If R is a relation on a set A then show that

(i) R is symmetric $\Rightarrow R^{-1}$ is symmetric and $R^{-1} = R$

(ii) R is transitive $\Rightarrow R^{-1}$ is transitive.

Ans By definition $R^{-1} = \{(y,x) : (x,y) \in R\}$

(i) Since R is symmetric, $(x,y) \in R \Rightarrow (y,x) \in R$

we need to show that $(a,b) \in R^{-1} \Rightarrow (b,a) \in R^{-1}$

Now $(a,b) \in R^{-1} \Rightarrow (b,a) \in R \xRightarrow{R \text{ symmetric}} (a,b) \in R \Rightarrow (b,a) \in R^{-1}$

R^{-1} is symmetric, $(a,b) \in R^{-1} \Rightarrow (a,b) \in R$ $\therefore R^{-1} = R$

(ii) Let $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1} \Rightarrow (b,a) \in R$ and $(c,b) \in R$

But R is transitive, so $(c,a) \in R \Rightarrow (a,c) \in R^{-1}$

Thus $(a,b) \in R^{-1}, (b,c) \in R^{-1} \Rightarrow (a,c) \in R^{-1}$

Hence R^{-1} is transitive.

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Q. Give examples of a relation which have the following properties

A) Let $A=\{1,2,3,4\}$, $A\times A=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

(i) Reflexive, transitive but not symmetric

$R_1=\{(1,1),(2,2),(3,3),(4,4),(2,3),(3,4),(2,4)\}$.

$(2,3),(3,4)\in R_1 \Rightarrow (2,4)\in R_1$, but $(2,3)\in R_1$ but $(3,2)\notin R_1$

(ii) symmetric but neither reflexive nor transitive $R_2=\{(2,2),(3,4),(4,3),(2,3),(3,2)\}$

$(3,4),(4,3),(2,3),(3,2)\in R_2$, $(1,1),(3,3),(4,4)\notin R_2$

$(2,3),(3,4)\in R_2$ but $(2,4)\notin R_2$

(iii) Reflexive, symmetric and transitive

$R_3=\{(1,1),(2,2),(3,3),(4,4),(2,4),(4,2)\}$

(iv) symmetric, transitive but not reflexive

$R_4=\{(2,3),(3,2),(3,3),(2,2)\}$

(v) Reflexive but neither symmetric nor transitive

$A=\{1,2,3\}$, $A\times A=\{(1,1),(2,2),(3,3),(1,2),(3,2),(1,3),(2,1),(2,3),(3,1)\}$,

$R_5=\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$

(vi) Neither Reflexive, nor symmetric nor transitive

Let $M\equiv$ set of all men, R denote the relation of 'father of'

a is not father of a, If a is a father of b then b cannot be father of a, If a is a father of b and b is father of c then a cannot be father of c