Blackbody Radiation-Section2

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1 Rayleigh-Jean's classical theory of blackbody radiation

Rayleigh and Jeans, two British physicists, attempted to explain the blackbody radiation curve theoretically, i.e. derived a mathematical expression for describing the blackbody radiation intensity I(λ) as function of wavelength λ at a xed temperature T. Their mathematical approach was based on classical wave theory, and was partially successful to reach its goal. However, their theory on blackbody radiation is signi cantly recognized.

1.1 Derivation: Rayleigh-Jean's law

Consider a cubical blackbody cavity of edge L. The inner wall of the cavity is coated with lamp black. At temperature T, the atoms and molecules on the inner walls of the cavity oscillate by thermal vibration. We shall call them simply as 'oscillators'. They perform simple harmonic oscillations. Electromagnetic waves (radiations) emit from these oscillators and get re ected and re-re ected on the inner walls of the cavity. These emitted waves along the length of the cube form standing waves under the condition that

$$L = n\frac{\lambda}{2}$$

where λ is wavelength of electromagnetic wave and hence $\lambda = \frac{2L}{n}$, where n are nonzero integers (n=1,2,3...etc.). Each

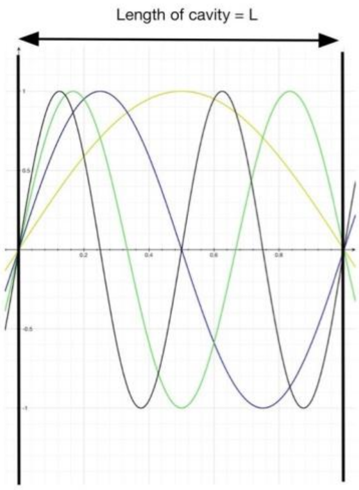

Figure 1: Di erent radiation modes inside a hot cavity

n value de nes a particular 'mode' of oscillation. Various such modes are shown in gure 1 where the yellow, blue, green and dark colored wavefunctions corresponds to n=1,2,3 and 4, respectively. The wavenumber k is de ned as $k = \frac{2\pi}{\lambda}$ and therefore may written as

$$k = \frac{2\pi}{\frac{2L}{n}} = \frac{\pi n}{L}$$

Considering the three-dimensional cartesian coordinate system, where n_x, n_y and n_z are integers along the x, y and z direction inside the cavity:

$$k_x = \frac{\pi n_x}{L_x}, k_y = \frac{\pi n_y}{L_y}, k_z = \frac{\pi n_z}{L_z}$$

Since we have considered a cube, L_x=L_y=L_z=L. Therefore,

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\pi^2(n_x^2 + n_y^2 + n_z^2)}{L^2}$$

as, $k = \frac{2\pi}{\lambda}$

$$n^2 = n_x^2 + n_y^2 + n_z^2 = \frac{k^2L^2}{\pi^2} = \frac{4L^2}{\lambda^2} \quad (1)$$ Hence,

$$n = \frac{2L}{\lambda}$$

di erentiating

$$\delta n = -\frac{2L}{\lambda^2}\delta\lambda \quad (2)$$

As mentioned earlier, n_x, n_y and n_z are di erent modes of oscillations. Corresponding to n_x, n_y and n_z, there exist k_x, k_y and k_z values respectively. If we consider a three dimensional momentum (k) space, its three axes are k_x, k_y and k_z, respectively. Now, we can transform this k-space into a n-space via rescaling the axes by multiplying them with $\frac{L_x}{\pi}$, $\frac{L_y}{\pi}$ and $\frac{L_z}{\pi}$ respectively. We may call this n-space as 'mode'-space. In gure 2, such a n-space is shown.

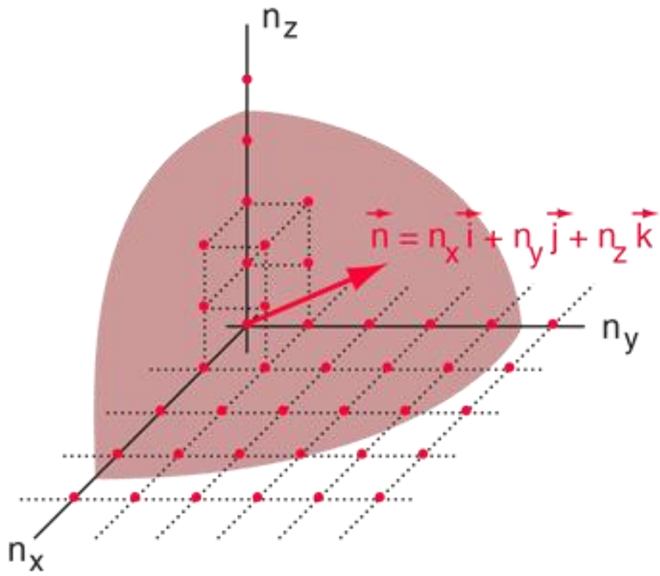

Figure 2: Schematic of a n-space

Within the rst octant (where n_x, n_y and n_z all are positive), volume of spherical shell in n-space within radius n and n+δn, i.e. number of modes:

$$\delta N = \frac{1}{8}4\pi n^2 \delta n$$

$$\delta N = \frac{1}{8}4\pi(n_x^2 + n_y^2 + n_z^2)\delta n$$

using eqn 1 and eqn 2,

$$\delta N = -\frac{1}{8}4\pi\frac{4L^2}{\lambda^2}\frac{2L}{\lambda^2}\delta\lambda$$

$$\frac{\partial N}{\partial \lambda} = -\frac{4\pi L^3}{\lambda^4} \quad (3)$$

The minus sign indicates that the number of modes decreases with increasing wavelength. Eqn (3) de nes the number of modes per unit wavelength. From the principle of equipartition of energy, each standing wave mode has average energy $k_B T$, where k_B is Boltzmann constant (k_B=1.38×10^{-23} J/K), and T is absolute temperature. Therefore, if E is the energy associated to each mode then,

$$\frac{\partial E}{\partial \lambda} = -\frac{4\pi k_B T L^3}{\lambda^4} \tag{4}$$

The energy density is de ned as,

$$\delta u = \frac{Energy(\delta E)}{Volume(L^3)}$$

, Therefore equation eqn (4) may be written as:

$$\frac{\partial u}{\partial \lambda} = \frac{1}{L^3}\frac{\partial E}{\partial \lambda} = -\frac{4\pi k_B T}{\lambda^4} \tag{5}$$

Now, interesting point to note is that, each oscillator can vibrate horizontally and vertically on the inner surface of the cavity, creating s-polarized and p-polarized electromagnetic waves. Therefore, the actual number of modes should be double than counted so far. The energy density should be double than as shown in eqn (5) which will be modi ed as,

$$\frac{\partial u}{\partial \lambda} = -\frac{8\pi k_B T}{\lambda^4} \tag{6}$$

Equation (6), is the Rayleigh-Jean's distribution law for blackbody radiation expressing the wavelength λ dependance of energy density u(λ). This distribution law may be expressed in terms of frequency ν. We know, $\lambda = \frac{c}{\nu}$, where c is the velocity of the electromagnetic waves. Di erentiating,

$$d\lambda = -\frac{c}{\nu^2}d\nu$$

Putting in equation (6),

$$\delta u = \frac{8\pi k_B T \nu^4}{c^4}\frac{c}{\nu^2}d\nu$$

$$\frac{\partial u}{\partial \nu} = \frac{8\pi k_B T \nu^4}{c^4}\frac{c}{\nu^2}$$

$$\frac{\partial u}{\partial \nu} = \frac{8\pi k_B T \nu^2}{c^3} \tag{7}$$

Equation (7) is the Rayleigh Jean's distribution law, that expresses energy density u as a function of frequency ν.

2 Rayleigh-Jean's distribution law: Success and Failure

Evidently, Rayleigh Jean's distribution law is very signi cant while discussing the phenomena of blackbody radiation. Equation (6) [or (7)] expresses the power law following which the energy density increases with decreasing wavelength [or, increasing frequency]. In gure 3, the experimental blackbody energy density distribution u(λ) is shown over a broad wavelength (λ) range, alongwith the Rayleigh Jean's distribution curve (dashed curve). The curve explains the distribution spectrum for higher values of λ. Unfortunately, the curve keeps on rising for lower λ values and therefore fails to produce a nite value as $u(\lambda) \longrightarrow \infty$ for $\lambda \longrightarrow 0$. This phenomena is called 'ultraviolet catastrophe'. Rayleigh Jean's law is unable to show that the energy density u(λ) of blackbody radiation decreases from the peak value for smaller λ. Therefore, Rayleigh Jean's law eqn (6) is successful for higher λ-side of the radiation spectrum and fails for the lower λ-side.

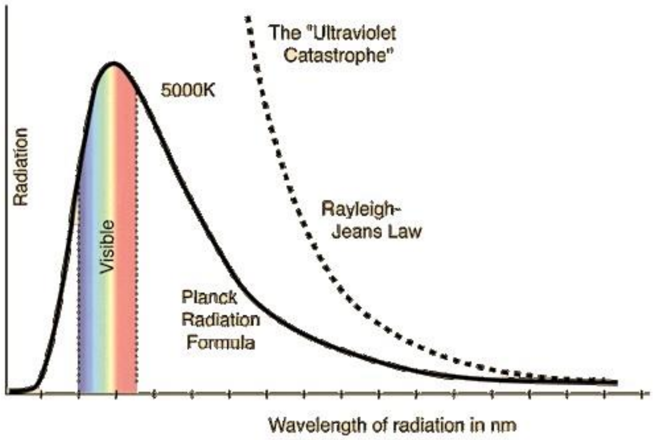

Figure 3: Rayleigh Jean's distribution function ts only for higher wavelength side of the blackbody radiation spectrum.

References

[1] Heat and Thermodynamics, Author:Zeemansky and Dittman, Publisher: Mc Graw Hill.

[2] Heat Thermodynamics and Statistical Physics, Author: Brij Lal,Subrahmanyam, Hemne. Publisher: S Chand.

[3] Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, Authors: R. Eisberg, R. Resnick, Publisher: Wiley

[4] Quantum Physics, Author: Stephen Gasiorowicz, Publisher: John Wiley and Sons.

[1]

[1]Figures are collected from online resources.