Q Express $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$, n being a positive integer.

A We know by De Moivre's theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, Expand by Binomial Theorem $\cos n\theta + i\sin n\theta = \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta + i^2\,{}^nC_2\cos^{n-2}\theta\sin^2\theta + i^3\,{}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots$

Equating real and imaginary parts on both sides

$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots$

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \cdots$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \cdots}$$

$$= \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \cdots}$$

Q To Expand $\cos^n\theta$ in terms of cosines of multiple angles, n being a positive integer

A Let $z = \cos\theta + i\sin\theta$, $\therefore \frac{1}{z} = \cos\theta - i\sin\theta$, $z + \frac{1}{z} = 2\cos\theta$

$z^n = \cos n\theta + i\sin n\theta$, $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$

$\Rightarrow z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$

and $(2\cos\theta)^n = \left(z + \frac{1}{z}\right)^n = z^n + {}^nC_1 z^{n-1}\frac{1}{z} + {}^nC_2 z^{n-2}\frac{1}{z^2} + \cdots + {}^nC_n\frac{1}{z^n}$

$\Rightarrow 2^n\cos^n\theta = \left(z^n + \frac{1}{z^n}\right) + {}^nC_1\left(z^{n-2} + \frac{1}{z^{n-2}}\right) + {}^nC_2\left(z^{n-4} + \frac{1}{z^{n-4}}\right) + \cdots$

$= 2\cos n\theta + 2\,{}^nC_1\cos(n-2)\theta + 2\cdot{}^nC_2\cos(n-4)\theta + \cdots$

Q To Expand $\sin^n\theta$ in terms of cosines of multiple angles, n being a positive even or odd integers.

A Let $z = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$, $z - \frac{1}{z} = 2i\sin\theta$

$z^n - \frac{1}{z^n} = 2i\sin n\theta$, $z^n + \frac{1}{z^n} = 2\cos n\theta$

Case I If n be positive even integer then

$(2i\sin\theta)^n = \left(z - \frac{1}{z}\right)^n = z^n - {}^nC_1 z^{n-1}\frac{1}{z} + {}^nC_2 z^{n-2}\frac{1}{z^2} - \cdots$

$\Rightarrow 2^n i^n\sin^n\theta = \left(z^n + \frac{1}{z^n}\right) - {}^nC_1\left(z^{n-2} + \frac{1}{z^{n-2}}\right) + \cdots$

$\therefore (-1)^{n/2}\,2^n\sin^n\theta = 2\cos n\theta - {}^nC_1\,2\cos(n-2)\theta + {}^nC_2\,2\cos(n-4)\theta - \cdots$

$(-1)^{n/2}\,2^{n-1}\sin^n\theta = \cos n\theta - n\cdot\cos(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\cos(n-4)\theta - \cdots$

last term is $\frac{1}{2}(-1)^{n/2}\,{}^nC_{n/2}$

Case II If n be a positive odd integer

$(2i\sin\theta)^n = (z-\frac{1}{z})^n = z^n - {}^nC_1 z^{n-2} + {}^nC_2 z^{n-4} - - - -$

$2^n (-1)^{\frac{n-1}{2}} \cdot i \cdot \sin^n\theta = (z^n - \frac{1}{z^n}) - {}^nC_1(z^{n-2} - \frac{1}{z^{n-2}}) + \dots$

$= 2i\sin n\theta - n \cdot 2i\sin(n-2)\theta + \frac{n(n-1)}{2} 2i\sin(n-4)\theta - - -$

$(-1)^{\frac{n-1}{2}} \cdot 2^{n-2}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{2}\sin(n-4)\theta$

Last term is $(-1)^{\frac{n-1}{2}}\ {}^nC_{\frac{n-1}{2}} \sin\theta$

Q To find $Log(\alpha+i\beta)$, where α, β are real

Ans Let $\alpha = r\cos\theta$, $\beta = r\sin\theta$ then $r^2 = \alpha^2+\beta^2$, $\tan\theta = \beta/\alpha$

$r = +\sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$, $\alpha+i\beta = r(\cos\theta+i\sin\theta) = re^{i\theta}$

$\therefore Log(\alpha+i\beta) = 2n\pi i + \log(re^{i\theta})$ | $\because e^z = e^{z+2n\pi i} = w$

$= 2n\pi i + \log r + i\theta$ general value | $Log\, w = z + 2n\pi i$

$= \frac{1}{2}\log(\alpha^2+\beta^2) + i(2n\pi + \tan^{-1}\beta/\alpha)$ | $z = \log w$

$n=0$, $\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\beta/\alpha$ Principal value

Q Find the general and Principal values of $Log(i)$

Ans $Log(i) = \log(0+i) = 2n\pi i + \log(i)$, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

$\log i = \frac{i\pi}{2}$ principal value, $\log(0+1) = \log 1 = 0$

$Log\, i = 2n\pi i + i\frac{\pi}{2} = \frac{1}{2}(4n+1)\pi i$

Q Express $(\alpha+i\beta)^{x+iy} = A+iB$

Ans $(\alpha+i\beta)^{(x+iy)} = e^{Log(\alpha+i\beta)^{(x+iy)}} = e^{(x+iy)Log(\alpha+i\beta)}$

Now $(x+iy)Log(\alpha+i\beta) = (x+iy)[\frac{1}{2}\log(\alpha^2+\beta^2) + i(2n\pi + \tan^{-1}\beta/\alpha)]$

$= \frac{x}{2}\log(\alpha^2+\beta^2) - y(2n\pi + \tan^{-1}\beta/\alpha) +$

$i[\frac{y}{2}\log(\alpha^2+\beta^2) + x(2n\pi + \tan^{-1}\beta/\alpha)] = p+iq$

$\therefore (\alpha+i\beta)^{(x+iy)} = e^{p+iq} = e^p\cos q + ie^p\cos q$

Q Find the general value i^i

Ans $i^i = e^{i\log i}$ Now $Log\, i = 2\pi n i + \log i = 2n\pi i + \frac{i\pi}{2}$

$\therefore i^i = e^{i\log i} = e^{i(4n+1)\frac{\pi i}{2}} = e^{-\frac{\pi}{2}(4n+1)}$

Q. If $\tan(x+iy) = u+iv$ PT $u^2+v^2+2u\cot 2x = 1$, $u^2+v^2 \neq 1$

Ans $\because \tan(x+iy) = u+iv \Rightarrow \tan(x-iy) = u-iv$

$\therefore \tan 2x = \tan[(x+iy)+(x-iy)] = \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$= 2u/(1-(u^2+v^2))$

$\therefore \cot 2x = \frac{1-u^2-v^2}{2u} \Rightarrow u^2+v^2-1 = -2u\cot 2x$

$\Rightarrow u^2+v^2+2u\cot 2x = 1$, $u^2+v^2 \neq 1$

Q. If $x = \log\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)$ PT $y = -i\log\tan\left(\frac{ix}{2}+\frac{\pi}{4}\right)$

A. we have $x = \log\tan\left(\frac{\pi}{4}+\frac{y}{2}\right) \Rightarrow e^{x} = \tan\left(\frac{\pi}{4}+\frac{y}{2}\right)$

$\Rightarrow \frac{e^{x/2}}{e^{-x/2}} = \frac{1+\tan y/2}{1-\tan y/2}$ $\Rightarrow$ Applying C & D, we get

$\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \frac{2}{2\tan y/2} = \cot y/2 = \frac{\cos y/2}{\sin y/2} = \frac{i(e^{iy/2}+e^{-iy/2})}{e^{iy/2}-e^{-iy/2}}$

$\Rightarrow \frac{e^{i\frac{ix}{2}}+e^{-i\frac{ix}{2}}}{-(e^{i\frac{ix}{2}}-e^{-i\frac{ix}{2}})} = i\frac{e^{iy/2}+e^{-iy/2}}{e^{iy/2}-e^{-iy/2}}$

$\Rightarrow \frac{2\cos ix/2}{-2i\sin ix/2} = \frac{\cot ix/2}{-i} = \frac{i(e^{iy/2}+e^{-iy/2})}{e^{iy/2}-e^{-iy/2}}$

$\frac{1}{\tan ix/2} = \frac{e^{iy/2}+e^{-iy/2}}{e^{iy/2}-e^{-iy/2}}$ Again Apply C & D

$\frac{1+\tan ix/2}{1-\tan ix/2} = \frac{2e^{iy/2}}{2e^{-iy/2}} = e^{iy} \Rightarrow e^{iy} = \tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)$

$\Rightarrow iy = \log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right) \Rightarrow y = -i\log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)$

Q. If $A+iB = c\tan(x+iy)$ PT $\tan 2x = \frac{2CA}{C^2-A^2-B^2}$

A. $\tan(x+iy) = \frac{A}{C}+\frac{iB}{C} \Rightarrow \tan(x-iy) = \frac{A}{C}-iB/_C$

$\tan 2x = \tan((x+iy)+(x-iy)) = \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$= \frac{2A}{C}\Big/\left(1-\left(\frac{A}{C}+iB/_C\right)\left(A/_C-iB/_C\right)\right)$

$= (2A/C)\Big/\left(1-\frac{A^2}{C^2}-\frac{B^2}{C^2}\right) = 2AC/(C^2-A^2-B^2)$

Q. If $i^{i^{i^{\cdots \text{to} \infty}}} = \alpha+i\beta$ PT $\tan\frac{\pi\alpha}{2} = \beta/\alpha$, $\alpha^2+\beta^2 = e^{-\pi\beta}$

A. $i^{i^{i^{\cdots \text{to} \infty}}} = \alpha+i\beta \Rightarrow \log(\alpha+i\beta) = i^{i^{i^{\cdots \text{to}\infty}}}\log i = (\alpha+i\beta)\log i$

$\log(\alpha+i\beta) = (\alpha+i\beta)\left(\frac{\pi}{2}i\right) = \frac{\pi}{2}\alpha i - \frac{\pi\beta}{2}$ Taking principal value

$\Rightarrow \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\beta/\alpha = \frac{-\pi\beta}{2} + \frac{\pi\alpha}{2}i$

separate real and imaginary parts

$\frac{1}{2}\log(\alpha^2+\beta^2) = -\frac{\pi\beta}{2} \Rightarrow \alpha^2+\beta^2 = e^{-\pi\beta}$ and

$\tan^{-1}\beta/\alpha = \frac{\pi\alpha}{2} \Rightarrow \tan\frac{\pi\alpha}{2} = \beta/\alpha$

Q8 State and Prove Gregory's Series (or) To express θ in ascending powers of $\tan\theta$ when $-\pi/4 \le \theta \le \pi/4$

Statement

Ans: If $-\pi/4 \le \theta \le \pi/4$ then

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots (-1)^{n-1}\frac{\tan^{2n-1}\theta}{2n-1} + \cdots \text{to } \infty$$

Proof: we know that $e^{i\theta} = \cos\theta + i\sin\theta = \cos\theta(1 + i\tan\theta)$

Taking log

$$i\theta = \log\{\cos\theta(1 + i\tan\theta)\} = \log\cos\theta + \log(1 + i\tan\theta)$$

$$\therefore i\theta = \log\cos\theta + i\tan\theta - \frac{1}{2}i^2\tan^2\theta + \frac{1}{3}i^3\tan^3\theta - \frac{1}{4}i^4\tan^4\theta + \cdots$$

$$= \left(\log\cos\theta + \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \cdots\right) + i\left(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots\right)$$

Equate imaginary parts

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots + (-1)^{n-1}\frac{\tan^{2n-1}\theta}{2n-1} + \cdots \text{to } \infty$$

where $-\frac{\pi}{4} \le \theta \le \pi/4$

This is called Gregory's Series.

Put $\tan\theta = x \Rightarrow \theta = \tan^{-1}x$ we get

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots \text{to } \infty$$

$-1 \le x \le 1 \quad \because -\pi/4 \le \theta \le \pi/4$

General form If $n\pi - \pi/4 \le \theta \le n\pi + \pi/4$ then

$$\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{1}{5}\tan^5\theta - \cdots \text{to } \infty$$

Put $\theta = \pi/4$ then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \text{to } \infty$$

$$= \frac{2}{1\cdot 3} + \frac{2}{5\cdot 7} + \frac{2}{9\cdot 11} + \cdots \text{to } \infty$$

$$\Rightarrow \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots \text{to } \infty = \pi/8$$

Also $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right) = \tan^{-1}1$

$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3}\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5} - \cdots \text{to } \infty\right) + \left(\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} \cdots \text{to } \infty\right)$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) + \cdots \text{to}$$

Q. Define Hyperbolic functions.

Ans. The function $\frac{e^x - e^{-x}}{2}$ and $\frac{e^x + e^{-x}}{2}$, where x is any number real or complex, are respectively called hyperbolic sine and hyperbolic cosine of x and are denoted by $\sinh x$ and $\cosh x$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x \,\mathrm{cosech}\, x = 1$

$\cosh x \,\mathrm{sech}\, x = 1$, $\tanh x \coth x = 1$, $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$e^x = \cosh x + \sinh x$, $e^{-x} = \cosh x - \sinh x$

$\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$, x is not angle

$\cosh^2 x - \sinh^2 x = 1$, $\mathrm{sech}^2 x + \tanh^2 x = 1$, $\mathrm{cosech}^2 x + 1 = \coth^2 x$

$\cosh(-x) = \cosh x$, $\sinh(-x) = -\sinh x$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$, $\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

$2\cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$

$2\sinh x \sinh y = \cosh(x+y) - \cosh(x-y)$

$2\sinh x \cosh y = \sinh(x+y) + \sin(x-y)$

$\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

$\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

$\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

Q. Relation between Hyperbolic and Circular functions

Ans. $\cosh x = \cos(ix)$, $\sinh x = -i\sin(ix)$, $\tanh x = -i\tan(ix)$

$\cos(ix) = \cosh x$, $\sin(ix) = i\sinh x$, $\tan ix = i\tanh x$

$\cosh ix = \cos x$, $\sin x = -i\sinh ix$, $\tan x = -i\tanh ix$

$\sinh^{-1} x = -i\sin^{-1}(ix)$, $\cosh^{-1} x = -i\cos^{-1} x$

$\tanh^{-1} x = -i\tan^{-1}(ix)$

Q. Find the value of inverse hyperbolic functions.

Ans. Let $y = \cosh^{-1} x \Rightarrow x = \cosh y = \frac{e^y + e^{-y}}{2} = \frac{e^y + \frac{1}{e^y}}{2}$

$\Rightarrow e^{2y} - 2xe^y + 1 = 0$ solving we get --

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1} \quad \because x-\sqrt{x^2-1} = \frac{1}{x+\sqrt{x^2-1}}$$

$\therefore e^y = x+\sqrt{x^2-1}$ or $(x+\sqrt{x^2-1})^{-1}$

$\therefore y = 2n\pi i \pm \log(x+\sqrt{x^2-1})$, $y = \log(x+\sqrt{x^2-1})$ principal value.

Let $y = \sinh^{-1}x \Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2+1} = x+\sqrt{x^2+1}$ or $-(x+\sqrt{x^2+1})^{-1}$

$\Rightarrow y = 2n\pi i + \log(x+\sqrt{x^2+1})$ or

$y = \log(-1) + 2n\pi i - \log(x+\sqrt{x^2+1})$

$= (2m+1)\pi i + 2n\pi i - \log(x+\sqrt{x^2+1})$

$\Rightarrow y = n\pi i + (-1)^n \log(x+\sqrt{x^2+1})$, $y = \log(x+\sqrt{x^2+1})$ principal value

Let $y = \tanh^{-1}x \Rightarrow x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$

$\Rightarrow e^{2y} = \frac{1+x}{1-x} \therefore 2y = 2n\pi i + \log\frac{(1+x)}{(1-x)}$

$y = \frac{1}{2} n\pi i + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$, $y = \frac{1}{2}\log\frac{(1+x)}{(1-x)}$ principal value

Q separate into real and imaginary parts

Ans (i) $\sin(\alpha+i\beta) = \sin\alpha\cos i\beta + \cos\alpha\sin i\beta = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta$

(ii) $\cos(\alpha+i\beta) = \cos\alpha\cos i\beta - \sin\alpha\sin i\beta = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

(iii) $\tan(\alpha+i\beta) = \frac{\sin(\alpha+i\beta)}{\cos(\alpha+i\beta)} = \frac{2\sin(\alpha+i\beta)\cos(\alpha-i\beta)}{2\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$

$= \frac{\sin 2\alpha + \sin 2i\beta}{\cos 2\alpha + \cos 2i\beta} = \frac{\sin 2\alpha + i\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$

(iv) $\cosh(\alpha+i\beta) = \cos i(\alpha+i\beta) = \cos(-\beta+i\alpha)$

$= \cos\beta\cos i\alpha + \sin i\alpha\sin(-\beta) = \cosh\alpha\cos\beta + i\sinh\alpha\sin\beta$

(v) $\sinh(\alpha+i\beta) = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$

(vi) $\tanh(\alpha+i\beta) = \frac{\sinh 2\alpha + i\sin 2\beta}{\cosh 2\alpha + \cos 2\beta}$

(vii) $\operatorname{sech}(\alpha+i\beta) = \frac{2\cosh\alpha\cos\beta - i\,2\sinh\alpha\sin\beta}{\cosh 2\alpha + \cos 2\beta}$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$\because x-\sqrt{x^2-1} = \frac{1}{x+\sqrt{x^2-1}}$

$\therefore e^y = x+\sqrt{x^2-1}$ or $(x+\sqrt{x^2-1})^{-1}$

$\therefore y = 2n\pi i \pm \log(x+\sqrt{x^2-1})$, $y = \log(x+\sqrt{x^2-1})$ Principal Value.

Let $y = \sinh^{-1}x \Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2+1} = x+\sqrt{x^2+1}$ or $-(x+\sqrt{x^2+1})^{-1}$

$\Rightarrow y = 2n\pi i + \log(x+\sqrt{x^2+1})$ or

$y = \log(-1) + 2n\pi i - \log(x+\sqrt{x^2+1})$

$= (2m+1)\pi i + 2n\pi i - \log(x+\sqrt{x^2+1})$

$\Rightarrow y = n\pi i + (-1)^n \log(x+\sqrt{x^2+1})$, $y = \log(x+\sqrt{x^2+1})$ Principal Value

Let $y = \tanh^{-1}x \Rightarrow x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$

$\Rightarrow e^{2y} = \frac{1+x}{1-x} \therefore 2y = 2n\pi i + \log\frac{(1+x)}{(1-x)}$

$y = \frac{1}{2} n\pi i + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$, $y = \frac{1}{2}\log\frac{(1+x)}{(1-x)}$ Principal Value

Q Separate into real and imaginary parts

Ans (i) $\sin(\alpha+i\beta) = \sin\alpha\cos i\beta + \cos\alpha\sin i\beta = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta$

(ii) $\cos(\alpha+i\beta) = \cos\alpha\cos i\beta - \sin\alpha\sin i\beta = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

(iii) $\tan(\alpha+i\beta) = \frac{\sin(\alpha+i\beta)}{\cos(\alpha+i\beta)} = \frac{2\sin(\alpha+i\beta)\cos(\alpha-i\beta)}{2\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$

$= \frac{\sin 2\alpha + \sin 2i\beta}{\cos 2\alpha + \cos 2i\beta} = \frac{\sin 2\alpha + i\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$

(iv) $\cosh(\alpha+i\beta) = \cos i(\alpha+i\beta) = \cos(-\beta+i\alpha)$

$= \cos\beta\cos i\alpha + \sin i\alpha\sin(-\beta) = \cosh\alpha\cos\beta + i\sinh\alpha\sin\beta$

(v) $\sinh(\alpha+i\beta) = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$

(vi) $\tanh(\alpha+i\beta) = \frac{\sinh 2\alpha + i\sin 2\beta}{\cosh 2\alpha + \cos 2\beta}$

(vii) $\operatorname{sech}(\alpha+i\beta) = \frac{2\cosh\alpha\cos\beta - i\,2\sinh\alpha\sin\beta}{\cosh 2\alpha + \cos 2\beta}$

Q. Express $\tan^{-1}(x+iy)$ in the form $A+iB$

Ans Let $\tan^{-1}(x+iy) = A+iB \Rightarrow \tan^{-1}(x-iy) = A-iB$

$\therefore \tan(A+iB) = x+iy, \ \tan(A-iB) = x-iy$

$\therefore \tan 2A = \tan((A+iB)+(A-iB)) = \frac{2x}{1-x^2-y^2}$

$\Rightarrow 2A = \tan^{-1}\frac{2x}{1-x^2-y^2} \quad \therefore A = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}$

Again

$\tan 2iB = \tan((A+iB)-(A-iB)) = \frac{2iy}{1+x^2+y^2}$

$i\tanh 2B = \frac{2iy}{1+x^2+y^2} \Rightarrow \tanh 2B = \frac{2y}{1+x^2+y^2}$

$\therefore B = \frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}$

Q. Express $\cos^{-1}(x+iy)$ in the form $A+iB$

A. Let $\cos^{-1}(x+iy) = A+iB \Rightarrow \cos^{-1}(x-iy) = A-iB$

$2A = \cos^{-1}(x+iy) + \cos^{-1}(x-iy)$

$= \cos^{-1}[(x+iy)(x-iy) - \sqrt{1-(x+iy)^2}\sqrt{1-(x-iy)^2}]$

$A = \frac{1}{2}\cos^{-1}[x^2+y^2 - \sqrt{(1-x^2+y^2)^2+4x^2y^2}$

$2iB = \cos^{-1}(x+iy) - \cos^{-1}(x-iy)$

$= \cos^{-1}[(x+iy)(x-iy) + \sqrt{1-(x+iy)^2}\sqrt{1-(x-iy)^2}]$

$\cos 2iB = x^2+y^2 + \sqrt{(1-x^2+y^2)^2+4x^2y^2}$

$\cosh 2B = x^2+y^2 + \sqrt{(1-x^2+y^2)^2+4x^2y^2}$

$\Rightarrow B = \frac{1}{2}\cosh^{-1}[x^2+y^2 + \sqrt{(1-x^2+y^2)^2+4x^2y^2}$

Q If $\cos^{-1}(u+iv) = \alpha+i\beta$, Prove that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $x^2-(1+u^2+v^2)x+u^2=0$

Ans We have $\cos^{-1}(u+iv) = \alpha+i\beta \Rightarrow \cos(\alpha+i\beta) = u+iv$

$\Rightarrow u+iv = \cos\alpha\cos i\beta - \sin\alpha\sin i\beta = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

$\Rightarrow u = \cos\alpha\cosh\beta, \ v = -\sin\alpha\sinh\beta$

$1+u^2+v^2 = 1+\cos^2\alpha\cosh^2\beta+\sin^2\alpha\sinh^2\beta = \cos^2\alpha+\cosh^2\beta$

The roots of required equation are $\cos^2\alpha$ and $\cosh^2\beta$

Hence equation is

$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

$\cancel{x^2-(1+u^2}$ $x^2-(\cos^2\alpha+\cosh^2\beta)x + \cos^2\alpha\cosh^2\beta = 0$

$x^2-(1+u^2+v^2)x+u^2 = 0$

$$(d)\quad \coth(x+iy) = \frac{\cosh(x+iy)}{\sinh(x+iy)} = \frac{\cos i(x+iy)}{\frac{1}{i}\sin i(x+iy)} = i\,.\,\frac{\cos(ix-y)}{\sin(ix-y)}$$

$$= i\,.\,\frac{2\sin(ix+y)\cos(ix-y)}{2\sin(ix+y)\sin(ix-y)} = i\,.\,\frac{\sin 2ix + \sin 2y}{\cos 2y - \cos 2ix}$$

$$= i\,.\,\frac{i\sinh 2x + \sin 2y}{\cos 2y - \cosh 2x} = \frac{-\sinh 2x}{\cos 2y - \cosh 2x} + i\,.\,\frac{\sin 2y}{\cos 2y - \cosh 2x}$$

$$= \frac{\sinh 2y}{\cosh 2x - \cos 2y} - i\,.\,\frac{\sin 2y}{\cosh 2x - \cos 2y}.$$

$$(e)\quad \text{sech}(x+iy) = \frac{1}{\cosh(x+iy)} = \frac{1}{\cos i(x+iy)}$$

$$= \frac{1}{\cos(ix-y)} = \frac{2\cos(ix+y)}{2\cos(ix+y)\cos(ix-y)} = \frac{2(\cos ix\cos y - \sin ix\sin y)}{\cos 2ix + \cos 2y}$$

$$= \frac{2(\cosh x\cos y - i\sinh x\sin y)}{\cosh 2x + \cos 2y}$$

$$= \frac{2\cosh x\cos y}{\cosh 2x + \cos 2y} - i\,.\,\frac{2\sinh x\sin y}{\cosh 2x + \cos 2y}.$$

$$(f)\quad \text{cosech}(x+iy) = \frac{1}{\sinh(x+iy)} = \frac{1}{\frac{1}{i}\sin i(x+iy)} = \frac{i}{\sin(ix-y)}$$

$$= i\,.\,\frac{2\sin(ix+y)}{2\sin(ix+y)\sin(ix-y)}$$

$$= i\,.\,\frac{2(\sin ix\cos y + \cos ix\sin y)}{\cos 2y - \cos 2ix} = i\,.\,\frac{2(i\sinh x\cos y + \cosh x\sin y)}{\cos 2y - \cosh 2x}$$

$$= -\frac{2\sinh x\cos y}{\cos 2y - \cosh 2x} + i\,.\,\frac{2\cosh x\sin y}{\cos 2y - \cosh 2x}$$

$$= \frac{2\sinh x\cos y}{\cosh 2x - \cos 2y} - i\,.\,\frac{2\cosh x\sin y}{\cosh 2x - \cos 2y}.$$

Example 3. *If* $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, *then prove that*

(*i*) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$ (*ii*) $\cosh u = \sec\theta$. (M.D.U. Dec. 2012)

Sol.
$$u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

(*i*)
$$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$\Rightarrow \quad e^{u/2}\,.\,e^{u/2} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} \quad \Rightarrow \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

By componendo and dividendo

$$\frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \frac{\left(1+\tan\frac{\theta}{2}\right)-\left(1-\tan\frac{\theta}{2}\right)}{\left(1+\tan\frac{\theta}{2}\right)+\left(1-\tan\frac{\theta}{2}\right)} \quad \Rightarrow \quad \tanh\frac{u}{2} = \tan\frac{\theta}{2}$$

(*ii*)
$$\cosh u = \frac{1+\tanh^2\frac{u}{2}}{1-\tanh^2\frac{u}{2}} = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \qquad \text{[Using part } (i)]$$

$$= \frac{1}{\cos\theta} = \sec\theta.$$

Example 4. *If $\sin(A+iB) = x+iy$, prove that*

(*i*) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (*ii*) $x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1.$

Sol. $x + iy = \sin(A + iB)$

$= \sin A \cos iB + \cos A \sin iB = \sin A \cosh B + i \cos A \sinh B$

Equating real and imaginary parts on both sides

$$x = \sin A \cosh B;\ y = \cos A \sinh B \qquad \ldots(i)$$

From (*i*), $\dfrac{x}{\cosh B} = \sin A;\ \dfrac{y}{\sinh B} = \cos A$

Squaring and adding, $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A = 1$

Also from (*i*), $\dfrac{x}{\sin A} = \cosh B;\ \dfrac{y}{\cos A} = \sinh B$

Squaring and subtracting, $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1$

or $\quad x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1.$

Example 5. *If $x + iy = \cosh(u + iv)$ show that*

(*i*) $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$ (*ii*) $x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = 1.$

(M.D.U. Dec. 2010)

$[\because \quad \cosh\theta = \cos i\theta]$

Sol. $x + iy = \cosh(u + iv) = \cos i(u + iv)$

$= \cos(iu - v) = \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v$

Equating the real and imaginary parts $\qquad \ldots(i)$

$$x = \cosh u \cos v;\ y = \sinh u \sin v$$

From (*i*), $\dfrac{x}{\cosh u} = \cos v,\ \dfrac{y}{\sinh u} = \sin v$

Squaring and adding, $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v = 1$

From (i), $\dfrac{x}{\cos v} = \cosh u$; $\dfrac{y}{\sin v} = \sinh u$

Squaring and subtracting, $x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = \cosh^2 u - \sinh^2 u = 1$.

Example 6. *If $x + iy = \tan (A + iB)$; prove that*

(i) $x^2 + y^2 + 2x \cot 2A = 1$ (ii) $x^2 + y^2 - 2y \coth 2B + 1 = 0$.

Sol. $x + iy = \tan (A + iB)$

Changing i into $-i$, we get $x - iy = \tan (A - iB)$

Now $\tan 2A = \tan [(A + iB) + (A - iB)]$

$$= \frac{\tan (A + iB) + \tan (A - iB)}{1 - \tan (A + iB) \tan (A - iB)} = \frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)} = \frac{2x}{1 - (x^2 + y^2)}$$

or $\dfrac{1}{\cot 2A} = \dfrac{2x}{1 - (x^2 + y^2)}$ or $1 - (x^2 + y^2) = 2x \cot 2A$

or $x^2 + y^2 + 2x \cot 2A = 1$ ...(I)

Again $\tan (2iB) = \tan [(A + iB) - (A - iB)]$

$$= \frac{\tan (A + iB) - \tan (A - iB)}{1 + \tan (A + iB) \tan (A - iB)} = \frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)} = \frac{2iy}{1 + x^2 + y^2}$$

or $i \tanh 2B = \dfrac{2iy}{1 + x^2 + y^2}$ or $\dfrac{1}{\coth 2B} = \dfrac{2y}{1 + x^2 + y^2}$

or $1 + x^2 + y^2 = 2y \coth 2B$

Hence $x^2 + y^2 - 2y \coth 2B + 1 = 0$ ...(II)

Q To find the sum of sines of n angles which are in AP

A) Let the series be

$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+(n-1)\beta)$

Using the formula $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$2\sin\alpha \sin\beta/2 = \cos(\alpha-\beta/2) - \cos(\alpha+\beta/2)$

$2\sin(\alpha+\beta)\sin\beta/2 = \cos(\alpha+\beta/2) - \cos(\alpha+3\beta/2)$

$2\sin(\alpha+2\beta)\sin\beta/2 = \cos(\alpha+3\beta/2) - \cos(\alpha+5\beta/2)$

— — — — — —

$2\sin(\alpha+(n-1)\beta)\sin\beta/2 = \cos\{\alpha+(2n-3)\beta/2\} - \cos\{\alpha+\frac{1}{2}(2n-1)\beta\}$

Adding we get

$2S.\sin\beta/2 = \cos(\alpha-\beta/2) - \cos\{\alpha+(2n-1)\beta/2\}$

$= 2\sin(\alpha+(n-1)\beta/2)\sin n\beta/2$

$\therefore \quad S = \dfrac{\sin\{\alpha+(n-1)\beta/2\}\sin n\beta/2}{\sin\beta/2}$

Q To find the sum of cosines of n angles which are in AP

A) Let $C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos\{\alpha+(n-1)\beta\}$

Using the formula $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos\alpha\sin\beta/2 = \sin(\alpha+\beta/2) - \sin(\alpha-\beta/2)$

$2\cos(\alpha+\beta)\sin\beta/2 = \sin(\alpha+3\beta/2) - \sin(\alpha+\beta/2)$

$2\cos(\alpha+2\beta)\sin\beta/2 = \sin(\alpha+5\beta/2) - \sin(\alpha+3\beta/2)$

— — — — — —

$2\cos\{\alpha+(n-1)\beta\}\sin\beta/2 = \sin\{\alpha+(2n-1)\beta/2\} - \sin\{\alpha+(2n-3)\beta/2\}$

Adding we get

$2.C.\sin\beta/2 = \sin\{\alpha+(2n-1)\beta/2\} - \sin(\alpha-\beta/2)$

$= 2\cos\{\alpha+(n-1)\beta/2\}\sin n\beta/2$

$\therefore \quad C = \dfrac{\cos\{\alpha+(n-1)\beta/2\}\sin n\beta/2}{\sin\beta/2}$

Note: $\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha = \dfrac{\sin(n+1)\frac{\alpha}{2}\sin\frac{n\alpha}{2}}{\sin\alpha/2}$

$\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \dfrac{\cos(n+1)\alpha/2\,\sin\frac{n\alpha}{2}}{\sin\alpha/2}$

Q To Resolve $\sin\theta$ as an infinite product (or)
To show that $\sin\theta = \theta \prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$

Ans Schlomilch method: we have $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

$\sin A = 2\sin A/2 \sin\left(\frac{\pi}{2}+A/2\right) = 2\sin A/2 \sin\left(\frac{\pi+A}{2}\right)$ —— (1)

Then $\sin\theta = 2\sin\theta/2 \sin\frac{\pi+\theta}{2}$ —(2) put $\theta/2$ & $\frac{\pi+\theta}{2}$ for A successively in (1), we get

$\sin\theta/2 = 2\sin\frac{\theta}{2^2}\sin\left(\frac{\pi+\theta/2}{2}\right) = 2\sin\frac{\theta}{2^2}\sin\left(\frac{2\pi+\theta}{2^2}\right)$

& $\sin\left(\frac{\pi+\theta}{2}\right) = 2\sin\left(\frac{\pi+\theta}{2^2}\right)\sin\left(\frac{\pi+\frac{\pi+\theta}{2}}{2}\right) = 2\sin\frac{\pi+\theta}{2^2}\sin\left(\frac{3\pi+\theta}{2^2}\right)$

Now from (2), we get

$\sin\theta = 2^3\sin\frac{\theta}{2^2}\sin\left(\frac{\pi+\theta}{2^2}\right)\sin\left(\frac{2\pi+\theta}{2^2}\right)\sin\left(\frac{3\pi+\theta}{2^2}\right)$

Applying successively the formula (1). for these four factors, we get

$\sin\theta = 2^7\sin\frac{\theta}{2^3}\sin\left(\frac{\pi+\theta}{2^3}\right)\sin\left(\frac{2\pi+\theta}{2^3}\right)\sin\left(\frac{3\pi+\theta}{2^3}\right)\sin\left(\frac{4\pi+\theta}{2^3}\right)$
$\sin\left(\frac{5\pi+\theta}{2^3}\right)\sin\left(\frac{6\pi+\theta}{2^3}\right)\sin\left(\frac{7\pi+\theta}{2^3}\right)$

$= 2^{(2^3-1)}\sin\left(\frac{\theta}{2^3}\right)\sin\left(\frac{\pi+\theta}{2^3}\right)\cdots\sin\left(\frac{(2^3-1)\pi+\theta}{2^3}\right)$

$= 2^{(2^n-1)}\sin\left(\frac{\theta}{2^n}\right)\sin\left(\frac{\pi+\theta}{2^n}\right)\sin\left(\frac{2\pi+\theta}{2^n}\right)\cdots\sin\left(\frac{(2^n-1)\pi+\theta}{2^n}\right)$

$= 2^{p-1}\sin\left(\frac{\theta}{p}\right)\sin\left(\frac{\pi+\theta}{p}\right)\cdots\sin\frac{(p-2)\pi+\theta}{p}\sin\frac{(p-1)\pi+\theta}{p}$ —(3) , $p=2^n$

Last factor in (3) is $\sin\frac{(p-1)\pi+\theta}{p} = \sin\left(\pi-\frac{\pi-\theta}{p}\right) = \sin\left(\frac{\pi-\theta}{p}\right)$

2nd last factor in (3) is $\sin\frac{(p-2)\pi+\theta}{p} = \sin\left(\pi-\frac{2\pi-\theta}{p}\right) = \sin\left(\frac{2\pi-\theta}{p}\right)$ and so on

The $\left(\frac{p}{2}+1\right)^{th}$ factor from begining in (3) is

$\sin\frac{\left(\frac{p}{2}+1-1\right)\pi+\theta}{p} = \sin\frac{\left(\frac{p\pi}{2}+\theta\right)}{p} = \sin\left(\frac{\pi}{2}+\frac{\theta}{p}\right) = \cos\frac{\theta}{p}$

since p is even number, the number of factors in (3) is even

Thus combining the second and last factors, third and 2nd last, and so on leaving ~~the~~ 1st we get

$\sin\theta = 2^{p-1}\sin\left(\frac{\theta}{p}\right)\left\{\sin\frac{\pi+\theta}{p}\sin\frac{\pi-\theta}{p}\right\}\left\{\sin\frac{2\pi+\theta}{p}\sin\frac{2\pi-\theta}{p}\right\}\cdots$

$\cdots\times\left\{\sin\frac{\left(\frac{p}{2}-1\right)\pi+\theta}{p}\sin\frac{\left(\frac{p}{2}-1\right)\pi+\theta}{p}\right\}\cos\frac{\theta}{p}$

Now using $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$$\sin\theta = 2^{p-1}\sin\left(\frac{\theta}{p}\right)\left\{\sin^2\frac{\pi}{p} - \sin^2\frac{\theta}{p}\right\}\left\{\sin^2\frac{2\pi}{p} - \sin^2\frac{\theta}{p}\right\}\cdots$$
$$\cdots\left\{\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p} - \sin^2\frac{\theta}{p}\right\}\cos\theta/p \quad \text{--- (4)}$$

$$\Rightarrow \frac{\sin\theta}{\sin\frac{\theta}{p}} = 2^{p-1}\left\{\sin^2\frac{\pi}{p} - \sin^2\frac{\theta}{p}\right\}\left\{\sin^2\frac{2\pi}{p} - \sin^2\frac{\theta}{p}\right\}\cdots$$
$$\cdots\left\{\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p} - \sin^2\frac{\theta}{p}\right\}\cos\frac{\theta}{p} \quad \text{--- (5)}$$

Let $\theta \to 0$ then $\lim_{\theta\to 0}\frac{\sin\theta}{\sin\theta/p} = \lim_{\theta\to 0}\frac{\frac{\sin\theta}{\theta}\times\theta}{\theta/p \times \frac{\sin\theta/p}{\theta/p}} = p$

$\lim_{\theta\to 0}\cos\frac{\theta}{p} = 1$, $\lim_{\theta\to 0}\sin^2\frac{\theta}{p} = 0$

Taking limiting value of (5) we get

$$p = 2^{p-1}\sin^2\frac{\pi}{p}\sin^2\frac{2\pi}{p}\sin^2\frac{3\pi}{p}\cdots\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p} \quad \text{--- (6)}$$

Diving (4) by (6) we get

$$\sin\theta = p\sin\frac{\theta}{p}\left\{1 - \frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\pi}{p}}\right\}\left\{1 - \frac{\sin^2\theta/p}{\sin^2\frac{2\pi}{p}}\right\}\cdots$$
$$\cdots\left\{1 - \frac{\sin^2\theta/p}{\sin^2\left(\frac{p}{2}-1\right)\pi/p}\right\}\cos\theta/p \quad \text{--- (7)}$$

Let $p\to\infty$ then $\lim_{p\to\infty} p\sin\theta/p = \lim_{p\to\infty}\frac{\sin\theta/p}{\theta/p}\cdot\theta = \theta$

and $\lim_{p\to\infty}\frac{\sin^2\theta/p}{\sin^2\frac{r\pi}{p}} = \frac{\left(\lim_{p\to\infty}\frac{\sin\theta/p}{\theta/p}\right)^2}{\left(\lim_{p\to\infty}\frac{\sin\frac{r\pi}{p}}{\frac{r\pi}{p}}\right)^2}\cdot\frac{\theta^2}{r^2\pi^2} = \frac{\theta^2}{r^2\pi^2}$

and $\lim_{p\to\infty}\cos\frac{\theta}{p} = 1$

Hence
$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\cdots\text{to}\infty$$

$$\Rightarrow \sin\theta = \theta\prod_{r=1}^{\infty}\left(1 - \frac{\theta^2}{r^2\pi^2}\right)$$

Q. To express $\cos\theta$ as an infinite product (or)

To show that $\cos\theta = \prod_{r=1}^{\infty}\left(1 - \frac{4\theta^2}{(2r-1)^2\pi^2}\right)$

Ans: we know that $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sin\frac{A}{2}\sin\frac{\pi+A}{2}$

$\therefore \sin\theta = 2\sin\frac{\theta}{2}\sin\frac{\pi+\theta}{2}$, Replacing A by $\frac{\theta}{2}$ and $\frac{\pi+\theta}{2}$

$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}$, $\sin\frac{\pi+\theta}{2} = 2\sin\frac{\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$

$\therefore \sin\theta = 2^3\sin\frac{\theta}{2^2}\sin\frac{\pi+\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$

Now we get

$\sin\theta = 2^{(2^n-1)}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\sin\frac{2\pi+\theta}{2^n}\cdots\sin\frac{(2^n-1)\pi+\theta}{2^n}$

$= 2^{p-1}\sin\frac{\theta}{p}\sin\frac{\pi+\theta}{p}\sin\frac{2\pi+\theta}{p}\cdots\sin\frac{(p-1)\pi+\theta}{p}$, $p=2^n$

put $\frac{\pi}{2}+\theta$ for θ in above equation, we get

$\cos\theta = 2^{p-1}\sin\left(\frac{\pi+2\theta}{2p}\right)\sin\frac{3\pi+2\theta}{2p}\cdots\sin\frac{(2p-1)\pi+2\theta}{2p}$ — (1)

Last factor $\sin\frac{2p\pi-(\pi-2\theta)}{2p} = \sin\left(\pi - \frac{\pi-2\theta}{2p}\right) = \sin\frac{\pi-2\theta}{2p}$

2nd last factor

$\sin\frac{2p\pi-(3\pi-2\theta)}{2p} = \sin\left(\pi - \frac{3\pi-2\theta}{2p}\right) = \sin\frac{3\pi-2\theta}{2p}$

Combining by $\sin(A+B)\sin(A-B) = \sin^2A - \sin^2B$, we get

$\cos\theta = 2^{p-1}\left\{\sin^2\frac{\pi}{2p} - \sin^2\frac{2\theta}{2p}\right\}\left\{\sin^2\frac{3\pi}{2p} - \sin^2\frac{2\theta}{2p}\right\}\cdots$

$\lim \theta\to 0$ then $\lim_{\theta\to 0}\cos\theta = 1$, $\lim_{\theta\to 0}\sin^2\frac{2\theta}{2p} = 0$

$\therefore 1 = 2^{p-1}\sin^2\frac{\pi}{2p}\sin^2\frac{3\pi}{2p}\sin^2\frac{5\pi}{2p}\cdots$ — (2)

dividing (1) by (2) we get

$\cos\theta = \left\{1 - \frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{\pi}{2p}}\right\}\left\{1 - \frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{3\pi}{2p}}\right\}\cdots$

$\lim p\to\infty$ then $\lim_{p\to\infty}\left(\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{\pi}{2p}}\right) = 4\theta^2\left(\lim_{p\to\infty}\frac{\sin\frac{2\theta}{2p}}{\sin\frac{\pi}{2p}}\right)^2 = \frac{4\theta^2}{\pi^2}$

Hence

$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\cdots$

$\cos\theta = \prod_{r=1}^{\infty}\left(1 - \frac{4\theta^2}{(2r-1)^2\pi^2}\right)$

Note: $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\theta\prod\left(1 - \frac{\theta^2}{r^2\pi^2}\right)}{\prod\left(1 - \frac{4\theta^2}{(2r-1)^2\pi^2}\right)}$

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Q PT (i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$

(ii) $\sum\frac{1}{n^4}=\frac{\pi^4}{90}$

As we know that $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\cdots$

$\sin\theta=\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\cdots=\theta\left(1-\frac{\theta^2}{6}+\frac{\theta^4}{120}-\cdots\right)$

$\therefore \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots=\theta\left(1-\frac{\theta^2}{6}+\frac{\theta^4}{120}-\cdots\right)$

$\Rightarrow \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\cdots=1-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)$

Taking Logarithms of both sides, we get

$\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\cdots=\log\left[1-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)\right]$

$\Rightarrow \left(-\frac{\theta^2}{\pi^2}-\frac{1}{2}\frac{\theta^4}{\pi^4}-\cdots\right)+\left(-\frac{\theta^2}{2^2\pi^2}-\frac{1}{2}\frac{\theta^4}{2^4\pi^4}-\cdots\right)+\cdots$

$=-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)-\frac{1}{2}\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)^2-\cdots$

Equating the coefficients of θ^2 and θ^4 from both sides we get

$\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)=\frac{1}{6}\Rightarrow\sum\frac{1}{n^2}=\frac{\pi^2}{6}$

and $\frac{1}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right)=-\frac{1}{120}+\frac{1}{72}=\frac{1}{180}$

$\Rightarrow \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}$

Q PT (i) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$, (ii) $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96}$

As we know that $\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots$

and $\cos\theta=1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\cdots$

$\therefore \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots=1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\cdots$

Taking logarithms

$\log\left(1-\frac{4\theta^2}{\pi^2}\right)+\log\left(1-\frac{4\theta^2}{3^2\pi^2}\right)+\cdots=\log\left(1-\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\cdots\right)\right)$

$\Rightarrow \left(-\frac{4\theta^2}{\pi^2}-\frac{1}{2}\frac{16\theta^4}{\pi^4}-\cdots\right)+\left(-\frac{4\theta^2}{3^2\pi^2}-\frac{1}{2}\cdot\frac{16\theta^4}{3^4\pi^4}-\cdots\right)+\cdots=$

$-\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\cdots\right)-\frac{1}{2}\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\cdots\right)^2-\cdots$

Equating the coeffts of θ^2 and θ^4

$\frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)=\frac{1}{2}\Rightarrow\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$

and $\frac{8}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)=-\frac{1}{24}+\frac{1}{8}=\frac{1}{12}$

$\Rightarrow \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96}$